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DEVANEY CHAOS AND ENTROPY OF THE TRIDIAGONAL MARKOV CHAIN

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ABSTRACT

Tridiagonal Markov shifts with *m*- letters, $m \geq 3$, are defined and it is established here that the topological Markov chain on this Markov shift is Devaney chaotic(*DevC*). Further, discussing various general ideas about topological entropy for continuous maps on compact metric spaces we calculate the topological entropy of the tridiagonal Markov chain by the direct application of Perron-Frobenius theorem.

KEYWORDS: Markov shifts, Topological Markov chain, Topological Transitivity, SDIC, Devaney Chaos, Topological Entropy.

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I. INTRODUCTION

Chaos may be roughly defined as the disorderliness or irregularity witnessed in a time evolving process, called dynamical systems. In topological dynamical systems, only the topological aspects are studied as opposed to other aspects like differential or measure theoretic aspects. The principal focus in topological dynamics is topological conjugacy, a homeomorphism that commutes with self-maps on compact topological spaces that establishes the equivalence of two dynamical models from a topological point of view. It may be considered as a great tragedy in the study of dynamical systems that there is no royal road to determine chaos in the systems. Some methods found applicable and fruitful for a certain class may not be useful for the other classes. For example, sensitivity dependence on initial conditions (SDIC) has been considered as a signature of chaos. But in some systems it is not so much a pleasant business to show its existence even in a simple, as it seems, dynamical process. Calculation of Lyapunov exponents is another prominent way paved to tread firmly on the road to determine chaos. Further, in their much stated paper “*Period three implies chaos*” [1], Li and Yorke showed that the existence of period three points in an interval map ensures one that the map is chaotic. In this case too, determining period three points for most of the maps is not an easy job and also this is found inapplicable in higher dimensional models. Again, the time series analysis of data does not give a complete set of information about the dynamics of a given model. From these points of view, in a crude sense, the methods propounded till today may perhaps be called not self-sufficient.

The concept of entropy was first introduced in information theory by *Claude E. Shannon* in his 1948 paper entitled "*A Mathematical Theory of Communication*"[2]. After his name this entropy in information theory is usually known as *Shannon entropy*. Later on, in a 1958 paper, the concept of *metric entropy*, often called *Kolmogorov-Sinai entropy*, was introduced by Kolmogorov which was successfully developed by Sinai. Modeling on this concept of metric entropy, in 1965, Adler, Konheim and McAndrew [3] introduced the notion of *topological entropy* for topological spaces via covers. They projected it as an invariant of topological conjugacy and since then it grew to a useful tool for classifying dynamical systems according to topological conjugacy. Hereafter, Dinaburg in 1970 and Bowen [4] in 1971 independently introduced topological entropy in metric spaces. Bowen fruitfully proved the equivalence between the already existing notion of topological entropy for topological spaces and that defined for metric spaces.

Topological entropy is one of the simplest and most important quantities that gauge the complexity of a system. It is viewed as a measure of exponential growth rate of the number of distinguishable orbits of iterates or measure of expansiveness. After the definition given by Bowen and Dinaburg much has been done in the field and new concepts like *algebraic entropy* has come into the scenario. Many improved methods and algorithms have also been developed by this time. An algorithm for computing $h(f)$, the topological entropy for the self-map f, using kneading theory was presented for a continuous and unimodal map f in [5]. In [6], we have an improved algorithm for computing topological entropy using the same kneading theory. In paper [7], we have another effective method for this computation that uses the periodic points of the transformation. Computability of topological entropy is also a pertinent question in the theory of dynamical systems. This aspect has been dealt in a 2000 paper by Weirauch [8] and in a 2006 paper by Simonsen [9]. Spandl [10] calculated topological entropy of *S*-gap shifts along with computability conditions. In this paper we prove that the tridiagonal Markov chain is Devaney chaotic and calculate the topological entropy of this Markov chain by appropriately using the Perron-Frobenius theorem.

Our present paper is outlined as follows. In the upcoming section II, preliminary definitions are given, discussions on tridiagonal matrices, shift spaces and topological entropy are made and basic results related to these concepts and needed for our purposes have been reproduced. In section III, it has been established that the tridiagonal Markov chain is Devaney chaotic. Topological entropy of this Markov chain is calculated in section IV. Conclusion of our study has been made in section V.

II. PRELIMINARY DEFINITIONS, DISCUSSIONS AND BASIC RESULTS

2.1 Tridiagonal Matrices and Its Properties:

Band matrices [11], square matrices with non-zero entries in a band along the diagonals, occur largely in various applications mainly in the solution of physical problems. Such matrices arise and are extensively used in the solution to the steady-state heat flow problem for a plate. Tridiagonal matrices are band matrices with band width 3 which are widely used to estimate the unsteady conduction of heat in a rod when the temperatures at distinct points on the rod change with time. Here the temperature vectors are fruitfully expressed by using tridiagonal matrices and the solution to the problems has been resolved perfectly. It is quite reasonable to expect that the problems related to steady heat flow and unsteady conduction of heat may be studied more fruitfully and effectively and by this some new and more interesting results may be propounded by using the concepts and basic results of Markov shifts which correspond to the matrices used for these purposes.

A *tridiagonal matrix* is a band matrix with band width 3 where the non-zero entries appear only in the upper diagonal, main diagonal and lower diagonal. A fifth order tridiagonal matrix looks like:

$$\begin{bmatrix} a_1 & c_1 & 0 & 0 & 0 \\ b_1 & a_2 & c_2 & 0 & 0 \\ 0 & b_2 & a_3 & c_3 & 0 \\ 0 & 0 & b_3 & a_4 & c_4 \\ 0 & 0 & 0 & b_4 & a_5 \end{bmatrix} \text{ and denoted simply as } \begin{bmatrix} a_1 & c_1 & & & 0 \\ b_1 & a_2 & c_2 & & \\ & b_2 & a_3 & c_3 & \\ & & b_3 & a_4 & c_4 \\ 0 & & & b_4 & a_5 \end{bmatrix}$$

If an m^{th} order tridiagonal matrix is also ***Toeplitz***, i.e. a ***diagonal-constant*** matrix, with main diagonal elements equal to *a*, upper diagonal elements equal to *c* and lower diagonal elements equal to *b*, then it is generally denoted by $T_m(a,b,c)$ and in particular if $a = b = c$, then it is denoted by $T_m(a)$. Thus $T_m(1)$ is clearly a *transition* or 0-1 *Toeplitz matrix*. In this paper we will mainly focus on this matrix and on the Markov shift due to this matrix. Thus

$$T_m(1) = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & .. & 0 \\ 1 & 1 & 1 & 0 & 0 & .. & 0 \\ 0 & 1 & 1 & 1 & 0 & .. & 0 \\ 0 & 0 & 1 & 1 & 1 & .. & 0 \\ 0 & 0 & 0 & 1 & 1 & .. & 0 \\ .. & .. & .. & .. & .. & .. & .. \\ 0 & 0 & 0 & 0 & 0 & .. & 1 \end{bmatrix}_{m\times m} = \begin{bmatrix} 1 & 1 & & & & & 0 \\ 1 & 1 & 1 & & & & \\ & 1 & 1 & 1 & & & \\ & & 1 & 1 & 1 & & \\ & & & 1 & 1 & .. & \\ & & & & & .. & .. \\ 0 & & & & & .. & 1 \end{bmatrix}_{m\times m}$$

More formally $T_m(1) = B(say) = [B_{ij}]_{m\times m}$ is the matrix whose entries are given by

$$B_{ij} = 1 \quad for \quad |i-j| \le 1 \quad and \quad B_{ij} = 0 \quad for \quad |i-j| \ge 2$$

We note here that the first and last columns (or rows) of this matrix contain exactly two 1's and every other column (or row) contains exactly three 1's. The following results are important in the implementations of various dynamical properties of the topological Markov chain on the Markov shift due to the matrix $T_m(1) = B$. Before any further discussion we first recall the definition of irreducibility and aperiodicity of matrices.

Definition2.1.1: ***Irreducible and aperiodic matrices***: A square matrix $A = [A_{ij}]_{m\times m}$ is irreducible if for every $i, j \in \mathrm{N}, 1 \le i, j \le m$, $\exists n \in \mathrm{N}$ such that $A_{ij}^n > 0$ i.e. the $(i, j)^{th}$ entry of the matrix A^n is positive. A is *aperiodic* if for every $i, j \in \mathrm{N}, 1 \le i, j \le m$, $\exists n \in \mathrm{N}$ such that $A_{ij}^k > 0, \forall k \ge n$. From these definitions it is clear that ***an aperiodic matrix is always irreducible***.

Proposition: 2.1.1[12, 13]: *The eigenvalues of* $T_m(a,b,c)$ *are given by*

$$\lambda_k = a - 2\sqrt{bc}\cos\left(\frac{k\pi}{m+1}\right), \quad k = 1,2,3,....,m$$

2.2 ***Shifts, Shifts of Finite Type, Graphs, Vertex Shifts and Tridiagonal Markov Shift (TDMS)***

For a finite set *A* of *m* elements, the *full A-shift* [14], denoted by A^Z, is the set of all the bi-infinite sequences $x = \{x_i\}_{i=-\infty}^{\infty}$ of elements from *A*. We refer to *A* as the *alphabet* and its elements as *symbols* or *letters*. Any finite sequence of *k*-letters from the alphabet *A* is a *word* or a *block of length k over A* or simply a *k-block*. For analytical purposes the sequence $x = \{x_i\}_{i=-\infty}^{\infty}$ in A^Z is typically denoted by $.......x_{-3}x_{-2}x_{-1} \cdot x_0x_1x_2x_3.......$. The block $x_{-k}....x_{-3}x_{-2}x_{-1}x_0x_1x_2....x_k$ in *x*, usually denoted by $x_{[-k,k]}$, is the *central (2k+1)-block of x*. A^Z is a compact topological space in the product topology [15] having a basis consisting of *cylinders* $C_{-k,l}(a_{-k},.....,a_l)$ defined by

$$C_{-k,l}(a_{-k},.....,a_l) = \{x = (x_i)_{i=-\infty}^{\infty} \in A^Z : \ x_i = a_i, \forall i \ \ with - k \le i \le l\}$$

The map $\sigma : A^Z \to A^Z$ defined by $\sigma(x) = ...x_{-3}x_{-2}x_{-1}x_0 \cdot x_1x_2x_3.....$ is the *shift map* on the full shift A^Z. The shift map σ on the full *A*-shift is continuous and is a homeomorphism of A^Z [15]. For $\rho > 1$, the metric $d_\rho : A^Z \times A^Z \to R$ defined by

$$d_\rho(x, y) = d_\rho((x_i)_{i=-\infty}^{\infty}, (y)_{-\infty}^{\infty}) = \rho^{-k}, \ where \ \ k = \min\{|i| : x_i \ne y_i\}$$

Generates the product topology on A^Z [15]. Consequently, (A^Z, σ) is a topological dynamical system which is *Devaney* as well as *Auslender-Yorke* chaotic. Further, σ has *chaotic* and also has *modified weakly chaotic dependence on initial conditions* [16].

It is to be noted that if $\rho > 2m - 1$, then for any $\varepsilon = 1/\rho^n$ and $a = \{a_i\}_{i=-\infty}^{\infty} \in A^Z$, the cylinder $C_{-n,n}(a_{-n},...,a_n)$, *called a symmetric cylinder*, is nothing but the open ball $B_{d_\rho}(a, \varepsilon)$ which contains all the bi-infinite sequences having $a_{[-n,n]}$ as their central *(2n+1)-block* [17].

Shift spaces X, also called *subshifts*, are subsets of the full *A*-shift A^Z such that no block from a specific set *F* of some certain blocks appears in any sequence in *X*. Here the set *F* in this context is termed as *the collection of forbidden blocks*. The shift space with the forbidden collection *F* is generally denoted by X_F. If the collection *F* of forbidden blocks of a shift space X_F is finite, then it is a *shift of finite type* or a *Markov shift*. Markov shifts can be described by *transition matrices* or by associated *directed graphs* of these transition matrices. The connection between transition matrices and directed graphs is well known. These two concepts can be linked to Markov shifts. More precisely, a transition matrix or its associated directed graph give rise to a Markov shift which is known as a vertex shift corresponding to the transition matrix. The actual process of forming these Markov shifts is as follows:

For a transition matrix $A = (A_{ij})_{m\times m}$, the vertex shift or Markov shift determined by *A* is denoted by $\hat{X}_A$ or Σ_A and is defined by

$$\Sigma_A = \{x = (x_i)_{i=-\infty}^{\infty} \in A^Z : A_{x_i x_{i+1}} = 1, \ \forall i \in Z\}$$

It is a 1-*step* Markov shift [5] with $\left(m^2 - \sum_{i,j=1}^{m} A_{ij}\right)$ numbers of forbidden 2-blocks given by the collection $F = \{ij : 1 \le i, j \le m,\ A_{ij} = 0\}$. *Admissible cylinders* in the Markov shift of the type $C_{-k,l}(a_{-k},....., a_l)$ with $k, l \in \mathrm{N}$ satisfying $A_{a_i a_{i+1}} = 1, \forall -k \le i \le l$, and *admissible symmetric cylinders* of the type $C_{-k,k}(a_{-k},....., a_k)$ play a pivotal role in the discourses of the dynamical properties of the *topological Markov chain* σ_A, restriction to the shift map σ on the Markov shift Σ_A. We have the following important proposition which is frequently taken into account in the studies related to topological Markov chains.

Proposition: 2.2.1*: If $\sigma_A : \Sigma_A \to \Sigma_A$ is a topological Markov chain corresponding to the transition matrix A, then,*

(i) A is irreducible if and only if $\sigma_A : \Sigma_A \to \Sigma_A$ is topologically transitive.

(ii) If A is aperiodic, then $\sigma_A : \Sigma_A \to \Sigma_A$ is topologically mixing.

The Markov shift determined by the tridiagonal Toeplitz matrix $T_m(1) = T(say)$ is given by $\Sigma_T = \hat{X}_T = \{x = (x_i)_{i=-\infty}^{\infty} \in A^{\mathrm{Z}} : x_i \in A = \{1,2,3,...., m\},\ i \in \mathrm{Z}, |x_i - x_{i+1}| \le 1\}$. Its forbidden class is $F = \{ij : 1 \le i, j \le m,\ 2 \le |i - j| \le m - 1\}$. That is, a bi-infinite sequence $(x_i)_{i=-\infty}^{\infty} \in \Sigma_m$ will be a member of Σ_T, if x_{i+1} follows x_i only when their difference is 0 or 1.

2.3: *Topological Entropy*

Topological entropy of maps has been defined in various equivalent ways. Here, we will mainly deal with the definitions which are connected to the definitions of separated sets, spanning sets and to that of open covers. To deal with these entropies we need the concept of a very special type of metric d_n^f for a topological dynamical system (X, f) defined by

$$d_n^f(x, y) = \max_{0 \le k < n}\{d(f^n(x), f^n(y))\}$$ Where d is the underlying metric for *X*.

The open ball $B_{d_n^f}(x, \varepsilon) = \{y \in X : d_n^f(x, y) < \varepsilon\} = \{y \in X : d(f^k(x), f^k(y)) < \varepsilon, \forall 0 \le k < n\}$ with radius $\varepsilon > 0$ and centre *x* consists of all the points in *X* whose orbits up to time *n* stay $\varepsilon-$close to the orbit segment $O_x^n = \{x, f(x), f^2(x), f^3(x),...., f^n(x)\}$.

It is easy to see that

$$B_{d_n^f}(x, \varepsilon) = B_d(x, \varepsilon) \cap f^{-1}(B_d(f(x), \varepsilon)) \cap f^{-2}(B_d(f^2(x), \varepsilon)) \cap ... \cap f^{-n+1}(B_d(f^{n-1}(x), \varepsilon))$$

2.3.1 ***Separated Sets and Topological Entropy***

Let (X, f) be a topological dynamical system with underlying metric *d*. For $\varepsilon > 0$ and $n \in \mathrm{N}$, a set $S \subseteq X$ is an $(n,\varepsilon)-$ separated set if for every pair of distinct points $x, y \in S$, we have that $d_n^f(x,y) \geq \varepsilon$. That is, with a finite scale resolution ε, every pair of points in an $(n,\varepsilon)-$ separated set have trajectories which can be recognized as different in time *n*. Since (X, d) is compact, so it follows that there exists $(n,\varepsilon)-$ separated sets and every such set is finite and bounded above uniformly. $Sep(f, n, \varepsilon)$ denotes the maximal cardinality of an $(n,\varepsilon)-$ separated set. Now, for any $\varepsilon > 0$, the quantity $h_{top}(f,\varepsilon) = \limsup_{n\to\infty} \frac{1}{n} \log Sep(f, n, \varepsilon)$ gives the exponential growth rate of $Sep(f, n, \varepsilon)$. Then the topological entropy $h_{top}(f)$ of a given dynamical system (X, f) is defined by $h_{top}(f) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log Sep(f, n, \varepsilon)$. Thus $h_{top}(f,\varepsilon)$ is the exponential growth rate of maximum number of orbits of length *n* which are distinguishable with finite precision ε and $h_{top}(f)$ is the exponential growth rate of maximum number of orbits of length *n* which are distinguishable with finite but arbitrary precision ε.

2.3.2: ***Spanning Sets and Topological Entropy***

For $\varepsilon > 0$ and $n \in \mathrm{N}$, a set $S \subseteq X$ is $(n,\varepsilon)-$ spanning if for every $x \in X$ there is $y \in S$ such that $d_n^f(x,y) < \varepsilon$. It follows that a set $S \subseteq X$ is $(n,\varepsilon)-$ spanning if for the finite resolution $\varepsilon > 0$, any point in *X* can be approximated with a point in *S* whose orbit up to *n* unit of time is indistinguishable. Equivalently, $S \subseteq X$ is $(n,\varepsilon)-$ spanning if and only if $X = \bigcup_{y\in S} B_{d_n^f}(y,\varepsilon)$. $Span(f, n, \varepsilon)$ denotes the minimal cardinality of a $(n,\varepsilon)-$ spanning set in *X*. Just like considering exponential growth rate of $Sep(f, n, \varepsilon)$ for finite precision $\varepsilon > 0$ and then taking limit as $\varepsilon \to 0$, we can define topological entropy in an alternative way. Though it seems different, it turns out that the consequence of both of these concepts gives the same topological entropy. In this sense both are considered as equivalent definitions. The following theorem establishes this fact in a more accurate and precise way:

Theorem: 2.3.1[8]: *For a topological dynamical system* $f : X \to X$ *on the metric space* (*X*, *d*),

$$h_{top}(f) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log Span(f, n, \varepsilon)$$

Proof: The proof of this theorem is based on the following two results

(i) $Span(f, n, \varepsilon) \leq Sep(f, n, \varepsilon)$ and (ii) $Sep(f, n, 2\varepsilon) \leq Span(f, n, \varepsilon)$

Let us prove these results one after another. Suppose *S* be an $(n,\varepsilon)-$ separated set of maximal cardinality so that $Sep(f, n, \varepsilon) = Card(S)$. We claim that S is also a $(n,\varepsilon)-$ spanning set. For given $x \in X - S$, consider the set $S \bigcup \{x\}$. Now, S being a maximal $(n,\varepsilon)-$ separated set $S \bigcup \{x\}$ cannot be an $(n,\varepsilon)-$ separated set. So, there must exists a point $y \in S$ such that $d_n^f(x,y) < \varepsilon$. Otherwise, if $d_n^f(x,y) \geq \varepsilon$ for some $y \in S$, then $S \bigcup \{x\}$ will become an $(n,\varepsilon)-$ separated set. Thus for any given $x \in X - S$, there always exists a point $y \in S$ which is $\varepsilon - close$ to x w. r. t. the $d_n - metric$ as well as the $d - metric$. This amounts to conclude here that S is $(n,\varepsilon)-$ spanning. Since the cardinality of any minimal $(n,\varepsilon)-$ spanning set S cannot exceed the cardinality of any other $(n,\varepsilon)-$ spanning set, so we immediately have that

$$Span(f,n,\varepsilon) \leq Card(S) = Sep(f,n,\varepsilon)$$

To prove (ii), let us consider a minimal $(n,\varepsilon)-$ spanning set S so that $Span(f,n,\varepsilon) = Card(S)$. Since, S is $(n,\varepsilon)-$ spanning, so we have that $X = \bigcup_{y \in S} B_{d_n}(y,\varepsilon)$. Again consider that $\hat{S}$ be a maximal $(n,2\varepsilon)-$ separated set so that $Card(\hat{\hat{S}}) = Sep(f,n,2\varepsilon)$. Now, we show that no $d_n^f - ball$ mentioned above can contain two distinct points $s,t \in \hat{S}$. n separated set. If possible, let the two distinct points $s,t \in \hat{S}$ are contained in a single $d_n^f - ball\ B_{d_n^f}(y,\varepsilon)$ for some $y \in S$. Then, from triangular inequality, we have,

$d_n^f(s,t) \leq d_n^f(s,y) + d_n^f(y,t) < \varepsilon + \varepsilon = 2\varepsilon$, a contradiction to our assumption that $\hat{S}$ is $(n,2\varepsilon)-$ separated. Then, it immediately follows that the cardinality of $\hat{S}$ cannot be more than the number of open balls which is equal to the cardinality of S. So, we have,

$$Sep(f,n,2\varepsilon) = Card(\hat{S}) \leq Card(S) = Span(f,n,\varepsilon)$$

i.e, $Sep(f,n,2\varepsilon) \leq Span(f,n,\varepsilon)$

Proof of the theorem: By the above two results (i) and (ii), we clearly have that

$$Sep(f,n,2\varepsilon) \leq Span(f,n,\varepsilon) \leq Sep(f,n,\varepsilon)$$

Now, for any $n \in \mathrm{N}$, we also have that

$$\limsup_{n\to\infty} \frac{1}{n}[Sep(f,n,2\varepsilon)] \leq \limsup_{n\to\infty} \frac{1}{n}[Span(f,n,\varepsilon)] \leq \limsup_{n\to\infty} \frac{1}{n}[Sep(f,n,\varepsilon)]$$

Taking limit as $\varepsilon \to 0$, it amounts that the left as well as the right hand side of the above relation converge, by definition of $h_{top}(f)$, to $h_{top}(f)$ and hence ultimately we have that

$$h_{top}(f) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log Span(f,n,\varepsilon)$$

2.3.3: ***Topological Entropy via Covers***

Consider a topological dynamical system (X, f) on the compact metric space (X, d). Then, X being compact every open cover of *X* is reducible to a finite sub cover. So, it is sufficient to consider only the finite covers of *X*. For given $n\in \mathrm{N}$ and $\varepsilon > 0$, we consider a particular class of open covers of *X*. Consider a class $C_n^{\varepsilon} = \{B_{d_n}(x,\varepsilon/2) : x\in S \;\; where\ S\ is\ (n,\varepsilon/2)-spanning\}$. Thus for every $(n,\varepsilon/2)-spanning$ *set S*, we have a finite collection like C_n^{ε} of open balls with centers at every point of the spanning set *S* and $d_n^f -$ diameter less than $\varepsilon > 0$. By definition of a spanning set *S*, we have, $\bigcup_{x\in S} B_{d_n}(x,\varepsilon/2) = X$ and so C_n^{ε}'s are finite open covers of X with open sets having $d_n^f -$ diameter less than $\varepsilon > 0$. $Cov(f,n,\varepsilon)$ denotes the cardinality of a minimal finite open cover of X containing open sets each of which has $d_n^f -$ diameter less than $\varepsilon > 0$. Then we have the following important theorem which leads us to define topological entropy in another alternative way:

Theorem: 2.3.1: *For any topological dynamical system* $f : X \to X$ *on the metric space* (*X*, *d*),

$$h_{top}(f) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log Cov(f,n,\varepsilon)$$

The proof of this theorem is an immediate consequence of the following Lemma:

Lemma: 2.3.1: *For any topological dynamical system* $f : X \to X$ *on the metric space* (*X*, *d*),

$$Span(f,n,\varepsilon) \le Cov(f,n,\varepsilon) \le Span(f,n,\varepsilon/2)$$

Proof of the Lemma: To prove the first inequality, we first claim that if *C* is any cover of *X* containing open sets having $d_n^f -$ diameter less than $\varepsilon > 0$, then each open set *U* of this cover is contained in an open ball $B_{d_n^f}(x,\varepsilon)$ centered at a point $x\in U$. For, U being an open set with $d_n^f -$ diameter less than $\varepsilon > 0$ and $x\in U$, for every $y\in U$, we have that $d_n^f(x,y) < \varepsilon$. So, it immediately follows that $y\in B_{d_n^f}(x,\varepsilon)$ and hence $U \subseteq B_{d_n^f}(x,\varepsilon)$ which is our claim. Then clearly $\{B_{d_n^f}(x,\varepsilon) : x\in U,\, U\in C\}$ is an open cover of *X* with $d_n^f -$ balls having radius $\varepsilon > 0$ and cardinality $Card(C)$. Then the set *S* of all the centres of this new cover will definitely form an $(n,\varepsilon)-$ spanning set of *X* with cardinality $Card(S) = Card(C)$. Suppose we initially have chosen *C* to be a minimal open cover such that $Cov(f,n,\varepsilon) = Card(C)$. Then the corresponding $(n,\varepsilon)-$ spanning set *S* of *X* obtained by the above process is such that $Cov(f,n,\varepsilon) = Card(S)$. In this case by definition of $Span(f,n,\varepsilon)$, we have that

$$Span(f,n,\varepsilon) \le Card(S) = Cov(f,n,\varepsilon) \quad \text{(A)}$$

To prove the second inequality, let *S* be an $(n,\varepsilon/2)-$ spanning set of *X*. Then it gives a cover of open balls of *X* with $d_n -$ diameter less than $\varepsilon > 0$ and with cardinality *Card*(*S*). If S is a minimal $(n,\varepsilon/2)-$ spanning set of *X*, then $Span(f,n,\varepsilon/2) = Card(S)$. Suppose C is the open cover $C = \{B_{d_n}(x,\varepsilon/2) : x\in S\}$ corresponding to this $(n,\varepsilon/2)-$ spanning set *S* of *X*, then the minimal cardinality cannot exceed the cardinality $Card(S) = Span(f, n, \varepsilon/2)$ and so we have

$$Cov(f, n, \varepsilon) \le Span(f, n, \varepsilon/2) \quad \text{(B)}$$

Combining (A) and (B), we get $Span(f, n, \varepsilon) \le Cov(f, n, \varepsilon) \le Span(f, n, \varepsilon/2)$.

Proof of the theorem: By the above Lemma, we have that

$$Span(f, n, \varepsilon) \le Cov(f, n, \varepsilon) \le Span(f, n, \varepsilon/2)$$

So, for each $\varepsilon > 0$,

$$\limsup_{n\to\infty} \frac{1}{n} \log Span(f, n, \varepsilon) \le \limsup_{n\to\infty} \frac{1}{n} \log Cov(f, n, \varepsilon) \le \limsup_{n\to\infty} \frac{1}{n} \log Span(f, n, \varepsilon/2)$$

i.e. $\limsup_{n\to\infty} \frac{1}{n} \log Span(f, n, \varepsilon) \le \lim_{n\to\infty} \frac{1}{n} \log Cov(f, n, \varepsilon) \le \limsup_{n\to\infty} \frac{1}{n} \log Span(f, n, \varepsilon/2)$

For arbitrarily small precision $\varepsilon > 0$, taking limit as $\varepsilon \to 0$, the left as well as the right hand limits tends to $h_{top}(f)$ and hence by sandwich theorem we ultimately have that

$$h_{top}(f) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log Cov(f, n, \varepsilon) \ \blacksquare$$

Example: 2. 3.1: Consider the well known doubling map $f : R/\mathrm{Z} \to R/\mathrm{Z}$ which is defined by $f(x) = 2x(\mathrm{mod}1)$. For $n \in \mathrm{N}$ and $0 < \varepsilon < 1/4$, we first construct $(n, \varepsilon)-$ separated and spanning sets of *X*.

For, $0 < \varepsilon < 1/4$, we can find a $k \in \mathrm{N}$ such that $1/2^{k+1} < \varepsilon < 1/2^k$. Here, from our assumption $0 < \varepsilon < 1/4$ we clearly have that $k \ge 2$. Consider the set $S_n = \left\{ \frac{i}{2^n} : 0 \le i \le 2^n - 1 \right\}$, i.e. the set of all *dyadic fractions* with denominator 2^n. We now show that S_{n+k-1} is an $(n, \varepsilon)-$ separated set.

Let $x, y \in S_{n+k-1}, x \ne y,$ be arbitrary. We know that for the doubling map $f(x) = 2x(\mathrm{mod}1)$, and for any $r, s \in R/\mathrm{Z}$ with $d(r, s) < 1/4$, $d(f(r), f(s)) = 2d(r, s)$. So, if there exists $0 \le l < n-1$ such that $d(f^l(x), f^l(y)) \ge 1/4$ we are done. If $d(f^l(x), f^l(y)) < 1/4$, then we repeatedly apply the rule $d(f(r), f(s)) = 2d(r, s)$ for $(n-1)-$ times and ultimately will get

$$d(f^{n-1}(x), f^{n-1}(y)) = 2^{n-1} d(x, y)$$

Since $x, y \in S_{n+k-1}, x \ne y,$ we have $d(x, y) \ge 1/2^{n+k-1}$ and

$$d(f^{n-1}(x), f^{n-1}(y)) = 2^{n-1} d(x, y) \ge 2^{n-1}/2^{n+k-1} = 1/2^k > \varepsilon$$

This shows that S_{n+k-1} is an $(n,\varepsilon)-$ separated set.

Next we show that S_{n+k} is an $(n,\varepsilon)-$ spanning set of *X*.

Let $x \in [0,1)$ be arbitrary. Then there exists a dyadic interval $I_i = \left[\frac{i}{2^{n+k}}, \frac{i+1}{2^{n+k}}\right], 0 \le i < 2^{n+k}$ such that $x \in I_i$. Now, if we take $y = \frac{i}{2^{n+k}} \, or \, \frac{i+1}{2^{n+k}}$, then we get, $d(x,y) \le \frac{1}{2^{n+k}}$ and for $0 \le l < n-1$,

$$d(x,y) \le 1/2^{n+k} \Rightarrow d(f^l(x), f^l(y)) \le 2^l / 2^{n+k} \le 2^{n-1}/2^{n+k} = 1/2^{k+1} < \varepsilon$$

Hence S_{n+k} is an $(n,\varepsilon)-$ spanning set of *X*.

Finally we compute the topological entropy of the doubling map using these $(n,\varepsilon)-$ spanning set and $(n,\varepsilon)-$ separated set.

By definition of $Span(f,n,\varepsilon)$ and $Sep(f,n,\varepsilon)$, we have,

$$Span(f,n,\varepsilon) \le Card(S_{n+k}) \text{ and } Sep(f,n,\varepsilon) \ge Card(S_{n+k-1})$$

So, we have,

$$h_{top}(f) = \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log Span(f,n,\varepsilon) \le \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log Card(S_{n+k}) = \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log 2^{n+k} = \log 2$$

And

$$h_{top}(f) = \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log Sep(f,n,\varepsilon) \ge \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log Card(S_{n+k-1}) = \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log 2^{n+k-1} = \log 2$$

Now, $h_{top}(f) \le \log 2, h_{top}(f) \ge \log 2 \Rightarrow h_{top}(f) = \log 2$. ■

2.4: *Topological Entropy of Shifts and of Markov Chains*

Entropy in a dynamical system measures the dynamical complexity of the mapping that defines the system. For a shift, it not only measures the complexity of the shift space, but also measures its information capacity. The entropy of a shift space is a number which is invariant under conjugacy and behaves well under factor codes and products. *Perron-Frobenius theory* of nonnegative matrices is a very useful tool applied to compute the entropies of *irreducible shifts* of finite type and of sofic shifts. In [14], we have methods for decomposing a general shift of finite type into irreducible pieces and for computing the entropy of those shifts with the help of these irreducible parts. Below we formally give the definition of entropy of a shift space prior to any further discussion on entropy of a shift of finite type.

Definition: 2.4.1: *Entropy of a shift*: Let *X* be a shift space. The entropy of the shift space *X* is denoted by $h(X)$ and is defined by

$$h(X) = \lim_{n\to\infty}\frac{1}{n}\log|B_n(X)|$$

Where $B_n(X)$ denotes the set of all the $n-blocks$ appearing in the points in *X*. From this definition it is clear that the entropy of a shift is nothing but the growth rate of $n-blocks$ in *X*. The calculation of entropy of a shift is important because it directly gives the topological entropy of its corresponding shift map. The following proposition gives the exact relation between these two entropies:

Proposition [14, 17]: 2.4.1: *For a shift dynamical system* (X, σ_X), $h(X) = h_{top}(\sigma_X)$.

This proposition can be fruitfully applied to find the topological entropy of a Markov chain. For the calculation of entropy of a shift space, the following two theorems are extensively used. Precise proofs of these theorems are available in [14].

Theorem: 2.4.1[14]: [*Perron-Frobenius Theorem*]: *For an irreducible matrix* $A \neq 0$, *A has a positive eigenvector* v_A *with corresponding eigenvalue* $\lambda_A > 0$ *that is both algebraically and geometrically simple. If* μ *is another eigenvalue for A, then* $|\mu| \leq \lambda_A$. *Any positive eigenvector for A is a positive multiple of* v_A.

T heorem: 2.4.2[14]: (i) *If G is an irreducible graph, then h* (X_G) *=log* $\lambda_{A(G)}$.

(ii) *If X be an irreducible M-step shift of finite type and G is the essential graph for which* $X^{[M+1]} = X_G$ *, then h(X)=log* $\lambda_{A(G)}$.

Now we are in a position to establish the following important results.

III. DEVANEY CHAOS OF THE TRIDIAGONAL MARKOV SHIFTS

Theorem: 3.1: $T_m(1) = B$ *is irreducible as well as aperiodic*.

Proof: We first prove that B^2, is a band matrix of band width 5. For this we need to show that along with the central 3-diagonals of B^2, the diagonals above and below these also contains non-zero entries. More explicitly we need to show that $B_{ij} \neq 0, \forall\ 1 \leq i, j \leq m \ with\ |i-j| \leq 2.$

Here $T_m(1) = B$ being a tridiagonal Toeplitz matrix, by definition, we have,

$B_{ij} = 1 \neq 0, \forall\ 1 \leq i, j \leq m \ with\ |i-j| \leq 1$ and $B_{ij} = 0$ otherwise.

We have, $B_{ii}^2 = \sum_{j=1}^{m} B_{ij}.B_{ji} \geq B_{ii}B_{ii} = 1.1 = 1 > 0,\ \forall i = 1,2,3,....., m$

$$B_{i(i+1)}^2 = \sum_{j=1}^{m-1} B_{ij}.B_{j(i+1)} \geq B_{ii}B_{i(i+1)} = 1.1 = 1 > 0,\ \forall i = 1,2,3,....., m-1$$

$$B_{(i+1)i}^2 = \sum_{j=1}^{m-1} B_{(i+1)j}.B_{ji} \geq B_{(i+1)i}B_{ii} = 1.1 = 1 > 0,\ \forall i = 1,2,3,....., m-1$$

$$B^2_{i(i+2)} = \sum_{j=1}^{m-2} B_{ij}.B_{j(i+2)} \geq B_{i(i+1)}B_{(i+1)(i+2)} = 1.1 = 1 > 0,\ \forall i = 1,2,3,.....,m-2$$

$$B^2_{(i+2)i} = \sum_{j=1}^{m-2} B_{(i+2)j}.B_{ji} \geq B_{(i+2)(i+1)}B_{(i+1)i} = 1.1 = 1 > 0,\ \forall i = 1,2,3,.....,m-2$$

Further, we have,

$$B^2_{(i+k)i} = \sum_{j=1}^{m} B_{(i+k)j}.B_{ji} = B_{(i+k)1}B_{1i} + B_{(i+k)2}B_{2i} + ... + B_{(i+k)m}B_{mi} = 0,\ \forall i = 1,2,...,m-2\ and\ k \geq 3$$

and $B^2_{i(i+k)} = \sum_{j=1}^{m} B_{ij}.B_{j(i+k)} = B_{i1}B_{1(i+k)} + B_{i2}B_{2(i+k)} + ... + B_{im}B_{m(i+1)} = 0,\ \forall i = 1,2,....,m-2\ and\ k \geq 3$

From the above facts it follows that B^2 is a pentadiagonal matrix or a band matrix of band width 5. That is B^2 will have two more non-zero diagonals than B. In a similar way we can show that $B^k, 1 \leq k \leq m-2,$ is a band matrix of band width $2k+1$. Particularly, B^{m-1} is a positive matrix. Thus every tridiagonal Toeplitz matrix $T_m(1) = B$ is aperiodic and hence is irreducible. ■

Theorem: 3.2: *The tridiagonal Markov chain $\sigma_B : \Sigma_B \to \Sigma_B$ is topologically transitive and mixing.*

Proof: By proposition 2.2.1, we have that for a transition matrix *A*, $T_m(1) = B$ is topologically transitive if and only if *A* is irreducible. Since every entry in $T_m(1) = B$ is either 0 or 1, so it is a transition matrix. Also, by the above theorem, $T_m(1) = B$ is irreducible. Therefore, we immediately have that the corresponding topological Markov chain $\sigma_B : \Sigma_B \to \Sigma_B$ is topologically transitive.

Also, by the same proposition we have that if *B* is aperiodic, then the corresponding topological Markov chain $\sigma_B : \Sigma_B \to \Sigma_B$ is topologically mixing. Since by the above theorem $T_m(1) = B$ is aperiodic, so $\sigma_B : \Sigma_B \to \Sigma_B$ is topologically mixing. ■

Theorem: 3.3: *The set of all the periodic points of $\sigma_B : \Sigma_B \to \Sigma_B$ is dense in Σ_B.*

Proof: Consider an arbitrary point $x = (x_i)_{i=-\infty}^{\infty} = ...x_{-n}......x_{-3}x_{-2}x_{-1} \cdot x_0x_1x_2x_3.......x_n....... \in \Sigma_B$.

Then, for any given $\varepsilon > 0$, however small, we show that there is a periodic point $p \in P(\sigma_B)$ such that $d_\rho(x,p) < \varepsilon$. That is, whatever small $\varepsilon > 0$ may be, the $\varepsilon -$ neighbourhood of *x* always contains a point of $P(\sigma_B)$.

For fixed $\varepsilon > 0$ and $\rho > 1$, it is easy to find a positive integer $n \in \mathrm{N}$ such that $\rho^{-n} < \varepsilon$. Now, for the point $x = (x_i)_{i=-\infty}^{\infty} \in \Sigma_B$, we find out a periodic point $p \in P(\sigma_B)$ in the ε-neighbourhood of *x*. First we closely observe the point $x = (x_i)_{i=-\infty}^{\infty} \in \Sigma_B$. We know that $x = (x_i)_{i=-\infty}^{\infty} \in \Sigma_B$ when and only when $|x_i - x_{i\pm1}| = 0\, or\, 1, \forall i \in Z\ and\ x_i^{'s} \in \{1,2,3,...,m\}$. That is, the difference between any two consecutive symbols in a point in Σ_B is either 0 or 1. Now, consider the central $(2n+1)-$block $x_{[-n,n]}$ of the point $x = (x_i)_{i=-\infty}^{\infty} \in \Sigma_B$ and the letters x_{-n} and x_n. Then, there may arise two cases. Case I: Let $x_{-n} = x_n$. In this case we can construct the periodic point p by concatenating the fixed block $x_{[-n,n]}$ infinite number of times in both directions. Since, $x_{-n} = x_n$ it is always allowed. Also, since the central $(2n+1)-$blocks of x and p agree, so $d_\rho(x, p) \le \rho^{-n} < \varepsilon$. That is, in this case we can easily find out a point in the $\varepsilon -$neighbourhood of *x*.

Case II: Let $x_{-n} \ne x_n$. Then there may arise two sub cases: $x_{-n} > x_n$ or $x_{-n} < x_n$.

If $x_{-n} > x_n$, then we consider the word $w = x_{[-n,n]} a_1 a_2 a_3 ... a_k x_{-n}$ where $x_n a_1 a_2 a_3 ... a_k x_{-n}$ is a word of consecutive digits in $A = \{1,2,3,..., m\}$. Then by concatenating the fixed word *w* infinite number of times in both directions we get a periodic point *p* in the $\varepsilon -$neighbourhood of *x*.

If $x_{-n} < x_n$, then take the word $w' = x_{[-n,n]} b_1 b_2 b_3 ... b_k x_{-n}$ where $x_n b_1 b_2 b_3 ... b_k x_{-n}$ is a word of consecutive digits in $A = \{1,2,3,..., m\}$ in descending order. In this case also by concatenating the fixed word $w' = x_{[-n,n]} b_1 b_2 b_3 ... b_k x_{-n}$ infinite number of times in both directions we get a periodic point *p* in the $\varepsilon -$neighbourhood of *x*.

Thus in all cases we always have a periodic point $p \in P(\sigma_B)$ in the $\varepsilon -$neighbourhood *x*. So, it follows that $P(\sigma_B)$ is dense in Σ_B. ■

Theorem: 3.4: *The shift map* $\sigma_B : \Sigma_B \to \Sigma_B$ *has sensitive dependence on initial conditions with the sensitivity constant* $\delta = 1$.

Proof: For arbitrarily chosen $\varepsilon > 0$ and $x = (x_i)_{i=-\infty}^{i=\infty} \in \Sigma_B$, we show that there always exists a point $y = (y_i)_{i=-\infty}^{\infty} \in \Sigma_B$ in the ε-neighbourhood of *x* such that $x_{k+1} \ne y_{k+1}$ for some $k \in \mathrm{N}$. Let $N_\varepsilon(x)$ denotes the ε-neighbourhood of *x*. Then, for fixed $\rho > 2m-1$, there exists $n \in \mathrm{N}$ such that $\rho^{-n} \le \varepsilon < \rho^{1-n}$ and so clearly we have that $C_{-n,n}(x_{-n},....,x_n) = B_{d_\rho}(x, \rho^{-n}) \subseteq B_{d_\rho}(x, \varepsilon) = N_\varepsilon(x)$. Now there may arise two cases: Case I: $x_n = x_{n+1}$ and Case II: $x_n \ne x_{n+1}$.

In the first case we choose $y \in \Sigma_B$ in *such* a way that

$$y = (y_i)_{i=-\infty}^{\infty} = x_{[-\infty,-1]} \cdot x_{[0,n]} x_{n+1}^{*} x_{[n+2,\infty]}^{*}\ where\ x_{n+1}^{*} = x_n \pm 1, x_{n+i}^{*} = x_{n+i}, \forall i (\ge 2) \in \mathrm{N}$$

Now, $x = (x_i)_{i=-\infty}^{i=\infty} \in \Sigma_B \Rightarrow x_{[-\infty,-1]}, x_{[0,n]}, x_{[n+2,\infty]}^{*} = x_{[n+2,\infty]} \in B(\Sigma_B)$, the language of Σ_B

$$\Rightarrow y = (y_i)_{i=-\infty}^{\infty} = x_{[-\infty,-1]} \cdot x_{[0,n]} x_{n+1}^{*} x_{[n+2,\infty]}^{*} \in \Sigma_B$$

Here x and y agree at least in their $(2n+1)$ central blocks. So, clearly $d_\rho(x,y) \le \rho^{-n} < \varepsilon$ and hence $y \in B_{d_\rho}(x,\rho^{-n}) = C_{-n,n}(x_{-n},....,x_n) \subseteq B_{d_\rho}(x,\varepsilon) = N(x)$.

In the second case, we choose $y \in \Sigma_B$ in such a way that $y = (y_i)_{i=-\infty}^{\infty} = x_{[-\infty,-1]} \cdot x_{[0,n]} x^*_{n+1} x^*_{[n+2,\infty]}$, where $x^*_{n+1} = x_n$ and $x^*_{n+i} = x_{n+i}$ for $|x_n - x_{n+2}| \le 1$ and $x^*_{n+i} = x_{n+i} \pm 1$ for $|x_n - x_{n+2}| = 2$, $\forall i \ge 2$.

In both cases $\sigma_B^{n+1}(x) = ...x_{-n}....x_0....\cdot x_{n+1}x_{n+2}.....,\quad \sigma_B^{n+1}(y) = x_{[-\infty,-1]}x_{[0,n]} \cdot x^*_{n+1}x^*_{[n+2,\infty]}$, $x^*_{n+1} \ne x_{n+1}$

$$\Rightarrow \sigma_B^{n+1}(x) \ne \sigma_B^{n+1}(y) \qquad [\because (\sigma_B^{n+1}(x))_0 \ne (\sigma_B^{n+1}(y))_0]$$

$$\Rightarrow d_\rho(\sigma_B^{n+1}(x), \sigma_B^{n+1}(y)) = 1 (= \delta)$$

Thus there exists $\delta(=1)$ such that for any $x = (x_i)_{i=-\infty}^{\infty} \in \Sigma_B$ and any neighbourhood $N(x)$ of x, there exists $y = (y_i)_{i=-\infty}^{\infty} \in N(x)$ and $k(= n+1) \in \mathrm{N}$ with $d_\rho(\sigma_B^k(x), \sigma_B^k(y)) = 1(=\delta)$.

Hence $\sigma_B : \Sigma_B \to \Sigma_B$ has sensitive dependence on initial conditions. ■

Theorem: 3.5: *The tridiagonal Markov chain* $\sigma_B : \Sigma_B \to \Sigma_B$ *is Devaney chaotic.*

Proof: This is an immediate consequence of the theorems 3.2, 3.3 and 3.4.

IV. TOPOLOGICAL ENTROPY OF TRIDIAGONAL MARKOV CHAINS

To calculate the topological entropy of a Markov shift that corresponds to a transition matrix requires the eigenvalue of greatest modulus of that matrix.

By Proposition: 2.1.1, the eigenvalues of $T_m(a,b,c)$ are given by

$$\lambda_k = a - 2\sqrt{bc}\cos\left(\frac{k\pi}{m+1}\right), \quad k = 1,2,3,....,m$$

So, immediately we have that the eigenvalues of $T_m(1,1,1) = T_m(1)$ are given by

$$\lambda_k = 1 - 2\cos\left(\frac{k\pi}{m+1}\right), \quad k = 1,2,3,....,m$$

Again by Proposition: 2.4.1, for a shift dynamical system (X, σ_X), $h(X) = h_{top}(\sigma_X)$.

Now, we calculate the topological entropy of Σ_B where $B = T_m(1)$ for $m = 3,4$ and then finally generalize it.

4.2.1: *Calculation of Topological Entropy for* Σ_B *Where* $B = T_3(1)$

Consider the matrix $B = T_3(1) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

Here $|B - \lambda I_3| = \begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 1-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^3 - 2(1-\lambda)$

Now,

$$|B - \lambda I_3| = 0 \Rightarrow (1-\lambda)[(1-\lambda)^2 - 2] = (1-\lambda)(\lambda^2 - 2\lambda - 1) = 0$$
$$\Rightarrow \lambda = 1, 1+\sqrt{2}, 1-\sqrt{2} = 1 - 2\cos\left[\frac{k\pi}{4}\right],\ k = 1,2,3$$
$$\Rightarrow \lambda_{\max} = 1+\sqrt{2} = 1 - 2\cos\left[\frac{3\pi}{4}\right]$$

Therefore, by Perron-Frobenius theorem,

$$h(\Sigma_B) = h_{top}(\sigma_B) = \log \lambda_{\max} = \log(1+\sqrt{2}) = \log[1 - 2\cos(3\pi/4)]\,.$$

4.2.2: *Calculation of Topological Entropy for* Σ_B *Where* $B = T_4(1)$

Consider the matrix tridiagonal Toeplitz matrix $B = T_4(1) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

$$|B - \lambda I_4| = \begin{vmatrix} 1-\lambda & 1 & 0 & 0 \\ 1 & 1-\lambda & 1 & 0 \\ 0 & 1 & 1-\lambda & 1 \\ 0 & 0 & 1 & 1-\lambda \end{vmatrix} = (1-\lambda)\cdot\begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 1-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix} - 1\cdot\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix}$$
$$= (1-\lambda).(1-\lambda)(\lambda^2 - 2\lambda - 1) - [(1-\lambda)^2 - 1]$$
$$= (1-\lambda)^2(\lambda^2 - 2\lambda - 1) - (1-\lambda)^2 + 1$$
$$= (\lambda^2 - 2\lambda)^2 - (\lambda^2 - 2\lambda) - 1$$
$$= \mu^2 - \mu - 1\,, \quad \textit{where } \mu = \lambda^2 - 2\lambda$$

Now,

$$\mu^2-\mu-1=0 \Rightarrow \mu=\lambda^2-2\lambda=\frac{1\pm\sqrt{5}}{2} \Rightarrow 2\lambda^2-4\lambda-(1\pm\sqrt{5})=0$$

$$\Rightarrow \lambda=1\pm\frac{1}{2}(\sqrt{5}\pm1)=1-2\cos\left(\frac{k\pi}{5}\right), k=1,2,3,4$$

Therefore, $\lambda_{\max}=1+\frac{1}{2}\left(\sqrt{5}+1\right)=1+2.\frac{1}{4}\left(\sqrt{5}+1\right)=1+2\cos\left[\frac{\pi}{5}\right]=1-2\cos\left[\frac{4\pi}{5}\right]$

and hence $h(\Sigma_B)=h_{top}(\sigma_B)=\log\lambda_{\max}=\log\left[1-2\cos\left[\frac{4\pi}{5}\right]\right]$. ■

4.2.3: *Topological Entropy for* Σ_B *Where* $B=T_m(1)$

Here

$$T_m(1)=\begin{bmatrix}1&1&0&0&0&..&0\\1&1&1&0&0&..&0\\0&1&1&1&0&..&0\\0&0&1&1&1&..&0\\0&0&0&1&1&..&0\\..&..&..&..&..&..&1\\0&0&0&0&0&..&1\end{bmatrix}_{m\times m}=\begin{bmatrix}1&1&&&&&0\\1&1&1&&&&\\&1&1&1&&&\\&&1&1&1&&\\&&&1&1&..&\\&&&&&..&1\\0&&&&&..&1\end{bmatrix}_{m\times m}$$

We have already mentioned that the eigenvalues of $B=T_m(1)$ are given by

$$\lambda_k=1-2\cos\left(\frac{k\pi}{m+1}\right),\quad k=1,2,3,....,m$$

We show that $\lambda_{\max}=1-2\cos\left[\frac{m\pi}{m+1}\right]=1-\cos\left[\pi-\frac{\pi}{m+1}\right]=1+\cos\left[\frac{\pi}{m+1}\right]$

We have, $0<\frac{\pi}{m+1}<\frac{2\pi}{m+1}<\frac{3\pi}{m+1}<\frac{4\pi}{m+1}<......<\frac{m\pi}{m+1}<\pi$

$$\Rightarrow \cos\left[\frac{\pi}{m+1}\right]>\cos\left[\frac{2\pi}{m+1}\right]>\cos\left[\frac{3\pi}{m+1}\right]>......>\cos\left[\frac{m\pi}{m+1}\right]$$

$$\Rightarrow 1-2\cos\left[\frac{\pi}{m+1}\right]<1-2\cos\left[\frac{2\pi}{m+1}\right]<1-2\cos\left[\frac{3\pi}{m+1}\right]<....<1-2\cos\left[\frac{m\pi}{m+1}\right]$$

$$\lambda_{\max}=1-2\cos\left[\frac{m\pi}{m+1}\right]=1-\cos\left[\pi-\frac{\pi}{m+1}\right]=1+\cos\left[\frac{\pi}{m+1}\right]$$

Hence we can conclude that $h(\Sigma_B) = h_{top}(\sigma_B) = \log \lambda_{\max} = \log\left[1 + 2\cos\left[\frac{\pi}{m+1}\right]\right]$. ■

V. CONCLUSIONS

In this paper we have mainly introduced tridiagonal shifts corresponding to the tridiagonal Toeplitz transition matrices which normally arise in the study of steady heat flow problems in a plate and also in unsteady conduction of heat on a rod. Detailed discussion on shifts and on entropies has been given in the preliminary discussion section. In the result part, we have examined the dynamical aspects of the Markov chain on the tridiagonal Markov shift in the theorems 3.1 to 3.5 where we have finally established that the tridiagonal Markov chain is *Devaney chaotic*. In section IV, we have calculated topological entropies for the Markov shifts with 3 and 4 letters and finally generalized it for *m* letters and found that the topological entropy is $\log\left[1 + 2\cos\left[\frac{\pi}{m+1}\right]\right] \to \log 3$ as $m \to \infty$. The positivity of topological entropy ensures the chaotic nature of the tridiagonal Markov chains. It is rationalize to expect that steady heat flow problems and also the problems related to unsteady conduction of heat on a rod can be more analytically and fruitfully interpreted, studied and resolved with these added concepts and results. Further it is expected that new but more simplified results/ideas will definitely come up and thereby open new dimensions in the study of thermal physics.

REFERENCES

1. Li, T. Y. and Yorke, J. A., *Period Three implies chaos*, The American Mathematical Monthly, vol. 82, No.10, pp. 985-992, 1975.
2. Shannon, C. E., *A Mathematical Theory of Communication*, Bell System Technical Journal, Vol.27, pp. 379-423, 623-656, July, October, 1948.
3. Adler, R., A. Konheim and M. McAndrew, *Topological Entropy*, Transactions of the American Mathematical Society 114, 1965, pp. 309-319.
4. Bowen, R., *Entropy for group endomorphisms and homogeneous spaces*, Transactions of the American Mathematical Society 153, 1971, pp. 401-414.
5. Collet, P. et. al., *Computing the topological entropy for maps*, Comm. Math. Phys. 88, 257-262, 1983.
6. Block, L. et al., *An improved algorithm for computing topological entropy*, Preprint, 1988.
7. Block L. and Coven E., *Approximating entropy of maps of the interval*, Dynamical Systems and Ergodic Theory, Banach Centre Publications, Polish Sci. publishers, Warsaw, pp. 237-242, 1989.
8. Weihrauch, K., "*Computable Analysis*", Springer-Verlag Berlin Heidelberg, New York, 2000.
9. Simonsen, J., *On the computability of topological entropy of subshifts,* Discrete Mathematics and Theoretical Computer Science 8,pp. 83-96,2006.
10. Spandl, Christoph, *Computing the topological entropy of shifts*, Electronic notes on Theoretical Computer Science 167, ELSEVIER, pp.131-155, 2007.

11. Lay, David C., *Linear Algebra and its Applications*, 3rd Edition, Pearson Education Inc. and Dorling Kindersley(India) Pvt. Ltd., 2011.
12. Kulkarni, Devadatta et al., *Eigenvalues of tridiagonal pseudo-Toeplitz matrices*, Linear Algebra and its Applications 297 (1999), pp. 63-80.
13. Borowska, Jolanta and Lacinska, Lena, *Eigenvalues of 2-Tridiagonal Matrix*, Journal of Applied Mathematics and Computational Mechanics 2015, 14(4), 11-17
14. Lind, Douglas and Markus, Brian, *An introduction to Symbolic Dynamics and Coding*, Cambridge University Press, Cambridge, U.K., 2003.
15. Brin, Michael and Stuck, Garret, *Introduction to Dynamical Systems*, Cambridge University Press, Cambridge, U.K., 2003.
16. Dutta, T.K. and Burhagohain, A., *Chaotic Aspects of the Shift Map on the Bi-sided Full m-shift*, International Journal of Applied Mathematics and Statistical Sciences, Vol.5, Issue 4, pp. 71-92, 2016.
17. Katok, A and Hasselblatt, B, *Introduction to the Modern Theory of Dynamical Systems*, Cambridge University Press, Cambridge, U.K., 1999.